Galois cohomology seminar
Week 7 - Background for Brauer groups

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Note on sources

The main source for these notes is Rapinchuk [?], with Milne as [?] as a secondary source. Goals to cover: background for Brauer groups, central simple algebras, double centralizer theorem, Wedderburn's theorem. A few things from Gille and Szamuely [?] were also used.

Contents

1 Central simple algebras

Throughout, we denote our base field by K. All K-algebras are assumed to be associative, unital, and finite-dimensional over K. For a unital algebra A with unit 1_A, we identify K with the subalgebra

$$\{x1_A : x \in K\}$$

so we can always think of K as embedded into A in this way.

1.1 Basic definitions and examples

Definition 1.1. A K-algebra A is **central** if the center of A is exactly K.

Definition 1.2. A K-algebra A is **simple** if it has no proper two sided ideals.

Example 1.1. Let D be a division algebra over K. Then D is clearly a simple K-algebra, since any nonzero element is a unit and generates all of D as an ideal. The center of D is a field, though not necessarily equal to K. We can at least say that D is a central simple algebra over $Z(D)$.

Example 1.2. Let A be any K-algebra. We will show that $M_n(A)$ is central. For $1 \leq i, j \leq n$, let $e_{ij} \in M_n(A)$ denote the matrix with a 1 in the ijth entry and zeros elsewhere. Note that for $X = (x_{ij}) \in M_n(A)$,

$$e_{ii}X = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ x_{i1} & \cdots & x_{in} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad Xe_{ii} = \begin{pmatrix} 0 & \cdots & x_{1i} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & x_{ni} & \cdots & 0 \end{pmatrix}$$

with the nonzero entries appearing in the ith row and ith column, respectively. Suppose $X \in M_n(D)$ is central, so $e_{ii}X = Xe_{ii}$ for $1 \leq i \leq n$. This forces all of the off-diagonal elements of X in the ith row and ith column to be zero. Hence X is diagonal.

Then since X commutes with permutation matrices, all the diagonal elements have to be the same. For example,

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathrm{Id} \end{pmatrix} X = \begin{pmatrix} 0 & x_{22} & \\ x_{11} & 0 & \\ & & * \end{pmatrix} = X \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} 0 & x_{11} & \\ x_{22} & 0 & \\ & & * \end{pmatrix}$$

Thus $X = \lambda \, \mathrm{Id}$ for some $\lambda \in K$, which shows that $M_n(A)$ is central.

Example 1.3. Let D be a division algebra over K. By the previous example, $M_n(D)$ is central. We also claim that is is simple. It suffices to show that for $X = (x_{ij}) \in M_n(D)$ nonzero, the two sided ideal $\langle X \rangle$ generated by X contains e_{ij} for all i, j, since the e_{ij} give a D-basis of $M_n(D)$. Because of the relation

$$e_{ki}e_{ij}e_{j\ell} = e_{k\ell}$$

if one e_{ij} lies in $\langle X \rangle$, then all of them do, so suffices to show that $e_{ij} \in \langle X \rangle$ for some i, j. Choose i, j so that $x_{ij} \neq 0$. Then

$$x_{ij}^{-1}e_{ii}Xe_{jj} = e_{ij}$$

so $e_{ij} \in \langle X \rangle$.

1.2 Wedderburn's theorem

The next goal is to prove Wedderburn's theorem, which says that all central simple algebras arise as $M_n(D)$ as in the previous example.

Theorem 1.1 (Wedderburn)**.** *Let A be a finite dimensional simple algebra over a field K. Then $A \cong M_n(D)$ for a unique $n \geq 1$ and a unique up to isomorphism division K-algebra D. Conversely, any algebra of the form $M_n(D)$ where D is a division algebra, is simple.*

Definition 1.3. Let A be a K-algebra. For A considered as a left A-module, we write ${}_AA$.

Remark 1.1. Let A be a K-algebra and M be an A-module. Then since $K \hookrightarrow A$, we can also view M as a K-module (aka K-vector space).

Lemma 1.2. *Let A be a (finite dimensional, unital, associative) simple K-algebra, and let $M \subset A$ be a minimal left ideal. Then*

1. *There exists $n > 0$ so that ${}_AA \cong \bigoplus_{i=1}^n M$ as A-modules.*
2. *Any A-module is iosmorphic to a direct sum of copies of M. In particular, M is the only simple A-module.*

Proof. Proposition 1 of Rapinchuk [**?**]. ☐

Lemma 1.3. *Let A be a K-algebra and let M be a left A-module. Then there is an isomorphism of K-algebras*

$$\mathrm{End}_A(M^n) \cong M_n(\mathrm{End}_A(M))$$

Proof. Stated and proved in somewhat more generality in Lemma 1 of Rapinchuk [**?**]. ☐

Lemma 1.4. *Let $A = M_n(D)$ where D is a division ring, and let $V = D^n$ be the space of n-columns on which A acts by left multiplication. Then V is a simple A-module and $\mathrm{End}_A(V) \cong D^{\mathrm{op}}$.*

Proof. Lemma 2 of Rapinchuk [**?**]. ☐

Now we finally prove Wedderburn's theorem.

Proof. First, we claim that

$$\mathrm{End}_A({}_AA) \to A^{\mathrm{op}} \qquad \phi \mapsto \phi(1)$$

is an isomorphism of K-algebras. If $\phi \in \mathrm{End}_A({}_AA)$, then for $a \in A$,

$$\phi(a) = a\phi(1)$$

so ϕ is determined by $\phi(1)$, so the claimed map is certainly bijective. It is K-linear because $K \hookrightarrow A$ and ϕ is A-linear. Finally, we show it is a homomorphism. We use $\cdot$ to denote multiplication in A^{op}. Then

$$\phi \circ \psi \mapsto \phi(\psi(1)) = \psi(1)\phi(1) = \phi(1) \cdot \psi(1)$$

so this establishes $\mathrm{End}_A({}_AA) \cong A^{\mathrm{op}}$ as K-algebras. By Proposition 1.2 part (1), ${}_AA \cong M^n$ as an A-module, so $\mathrm{End}_A({}_AA) \cong \mathrm{End}_A(M^n)$. By Lemma 1.3, we have $\mathrm{End}_A(M^n) \cong M_n(\mathrm{End}_A(M))$. Putting these isomorphisms together,

$$A^{\mathrm{op}} \cong \mathrm{End}_A({}_AA) \cong \mathrm{End}_A(M^n) \cong M_n(\mathrm{End}_A(M))$$

For any ring R, we have an isomorphism

$$M_n(R) \to M_n(R^{\mathrm{op}}) \qquad m \mapsto m^T$$

which in the case $R = \mathrm{End}_A(M)$, gives

$$M_n(\mathrm{End}_A(M))^{\mathrm{op}} \cong M_n(\mathrm{End}_A(M)^{\mathrm{op}})$$

so

$$A \cong (A^{\mathrm{op}})^{\mathrm{op}} \cong M_n(\mathrm{End}_A(M))^{\mathrm{op}} \cong M_n(\mathrm{End}_A(M)^{\mathrm{op}})$$

By Schur's lemma, $\mathrm{End}_A(M)$ is a division ring, so it's opposite is also a division algebra. Thus $A \cong M_n(D)$ for some division algebra D.

Now for uniqueness. Suppose $A \cong M_{n_1}(D_1) \cong M_{n_2}(D_2)$. Let $V_1 = D_1^{n_1}, V_2 = D_2^{n_2}$. By Lemma 1.4, V_1, V_2 are simple A-modules. Then by Proposition 1.2 part (2), $V_1 \cong V_2$ as A-modules. Using Lemma 1.4 again,

$$D_1^{\mathrm{op}} \cong \mathrm{End}_A(V_1) \cong \mathrm{End}_A(V_2) \cong D_2^{\mathrm{op}}$$

hence $D_1 \cong D_2$ as K-algebras, proving uniqueness of D. Also,

$$\dim_K A = n_1^2 \dim_K D_1 = n_2^2 \dim_K D_2$$

implies $n_1 = n_2$ since $D_1 \cong D_2$. □

1.3 Similarity of algebras

Lemma 1.5. *Let K be a field. Then*

1. *For any K-algebra R and positive integer n, $R \otimes_K M_n(K) \cong M_n(R)$.*
2. *For any positive integers m, n, $M_m(K) \otimes_K M_n(K) \cong M_{mn}(K)$.*

Proof. (1) An isomorphism is given by

$$R \otimes_K M_n(K) \to M_n(R) \qquad r \otimes x \mapsto rx$$

with inverse given by

$$M_n(R) \mapsto R \otimes_K M_n(K) \qquad (r_{ij}) \mapsto \sum_{i,j} r_{ij} \otimes e_{ij}$$

where e_{ij} is the matrix with 1 in the ijth entry and zeroes elsewhere.

(2) Up to choice of basis, $M_m(K) \cong \mathrm{End}_K(K^m)$, so we work with the endomorphism rings instead. There is a homomorphism

$$\mathrm{End}_K(K^m) \otimes_K \mathrm{End}_K(K^n) \to \mathrm{End}_K(K^m \otimes K^n) = \mathrm{End}_K(K^{mn})$$
$$\phi \otimes \psi \mapsto \Big(x \otimes y \mapsto \phi(x) \otimes \psi(y) \Big)$$

Note that by Proposition 2.4, the domain is a simple algebra. Then since the map is nonzero, it is injective (since the domain is simple). Then since the dimensions are equal, it is an isomorphism. □

Lemma 1.6 (Equivalent conditions for Brauer group equivalence)**.** *Let A_1, A_2 be central simple algebras over a field K, with $A_1 \cong M_{n_1}(D_1), A_2 \cong M_{n_2}(D_2)$ for unique integers n_1, n_2 and unique up to isomorphism division algebras D_1, D_2 (by Wedderburn's theorem 1.1). The following are equivalent.*

1. $D_1 \cong D_2$
2. *There exist integers m_1, m_2 such that $A_1 \otimes_K M_{m_1}(K) \cong A_2 \otimes_K M_{m_2}(K)$.*

Proof. First we prove (1) $\implies$ (2). Suppose $D_1 \cong D_2$. Then using Lemma 1.5 a few times,

$$\begin{aligned} A_1 \otimes_k M_{n_2}(K) &\cong M_{n_1}(D_1) \otimes_K M_{n_2}(K) \cong \Big(D_1 \otimes_K M_{n_1}(K) \Big) \otimes_K M_{n_2}(K) \\ &\cong D_1 \otimes_K \Big(M_{n_1}(K) \otimes_K M_{n_2}(K) \Big) \cong D_1 \otimes_K M_{n_1 n_2}(K) \\ &\cong M_{n_1 n_2}(D_1) \cong M_{n_1 n_2}(D_2) \cong D_2 \otimes_K M_{n_1 n_2}(K) \\ &\cong D_2 \otimes_K \otimes_K M_{n_2}(K) \otimes_K M_{n_1}(K) \cong A_2 \otimes_K M_{n_1}(K) \end{aligned}$$

which proves (2). For the converse, suppose $A_1 \otimes_K M_{m_1}(K) \cong A_2 \otimes_K M_{m_2}(K)$. Then using a similar chain of isomorphisms to the above,

$$M_{m_1 n_1}(D_1) \cong A_1 \otimes_K M_{m_1}(K) \cong A_2 \otimes_K M_{m_2}(K) \cong M_{m_2 n_2}(D_2)$$

By the uniqueness of Wedderburn's theorem 1.1, this implies $D_1 \cong D_2$. □

Definition 1.4. Two central simple K-algebras are **similar** if either of the two preceeding equivalent conditions holds, namely if their corresponding division algebras are isomorphic, or if they become isomorphic after tensoring with sufficiently large matrix algebras.

This is clearly an equivalence relation because of uniqueness in Wedderburn's theorem and because isomorphism is an equivalence relation.

Definition 1.5. As a set, the **Brauer group** of a field K, denoted $\mathrm{Br}(K)$, is similarity classes of central simple K-algebras.

2 Brauer group multiplication

Definition 2.1. The group operation for $\mathrm{Br}(K)$ is

$$[A] \cdot [B] = [A \otimes_K B]$$

Our goal for the rest of this section is to verify that this has the following properties.

1. It is well defined in the sense that the choice of representatives A,B don't matter.
2. It is well defined in the sense that $A \otimes_K B$ is central simple.
3. There is a unit.
4. Inverses exist.
5. It is associative.
6. It is commutative.

Items 5 and 6 are immediate, because $\otimes_K$ is associative and commutative. Item 3 is also immediate, since by definition, $A \sim A \otimes_K M_n(K)$, which is to say, $[M_n(K)]$ is the identity. We address item 1 first in section 2.1, then item 2 is addressed by sections 2.2 and 2.3. Finally, item 4 is addressed in section 2.4.

2.1 Independence of representatives

Lemma 2.1. *Assuming $A \otimes_K B$ is central simple, multiplication in $\mathrm{Br}(K)$ is independent of the choice of representatives.*

Proof. Suppose A', B' are other representatives with $[A] = [A'], [B] = [B']$. Then there are integers m, m', n, n' so that

$$A \otimes_K M_m(K) \cong A' \otimes_K M_{m'}(K) \qquad B \otimes_K M_n(K) \cong B' \otimes_K M_{n'}(K)$$

Then

$$(A \otimes_K B) \otimes_K M_{mn}(K) \cong (A' \otimes_K B') \otimes_K M_{m'n'}(K)$$

hence $[A \otimes_K B] = [A' \otimes_K B']$. □

2.2 $A \otimes_K B$ is central if A, B are central

Lemma 2.2. *Let V, W be K-vector spaces. Let $w_1, \ldots, w_n \in W$ be linearly independent. If there exist $v_1, \ldots, v_n \in V$ such that*

$$\sum_{i=1}^{n} v_i \otimes w_i = v_1 \otimes w_n + \cdots + v_n \otimes w_n = 0 \in V \otimes_K W$$

then $v_1 = \cdots = v_n = 0$.

Proof. Extend $w_1, \dots, w_n$ to a basis $w_1, \dots, w_n, \dots, w_{\dim W}$ of W. Let $x_1, \dots, x_{\dim V}$ be a basis of V, and write v_i as

$$v_i = \sum_j \alpha_{ij} x_j \qquad \alpha_{ij} \in K$$

Then

$$0 = \sum_i v_i \otimes w_i = \sum_i \left(\sum_j \alpha_{ij} x_j \right) \otimes w_i = \sum_{i,j} \alpha_{ij} (x_j \otimes w_i)$$

Since the simple tensors $x_j \otimes w_i$ form a basis of $V \otimes_K W$, by linear independence $\alpha_{ij} = 0$ for all i, j. That is, $v_i = 0$ for all i. □

Proposition 2.3 (Tensor product of central algebras is central)**.** *Let A, B be algebras over K. Then*

$$Z(A \otimes_K B) = Z(A) \otimes_K Z(B)$$

In particular, the tensor product of central algebras is central.

Proof. The inclusion $\supset$ is easy, so we dispatch it first. If $a \otimes b \in Z(A) \otimes Z(B)$, then for any $x \otimes y \in A \otimes B$,

$$(x \otimes y)(a \otimes b) = xa \otimes yb = ax \otimes by = (a \otimes b)(x \otimes y)$$

thus $a \otimes b \in Z(A \otimes B)$. The reverse inclusion is not so immediate. Let $z \in Z(A \otimes B)$, and write it as

$$z = \sum_{i=1}^{n} a_i \otimes b_i \qquad a_i \in A, b_i \in B$$

and choose this so that n is minimal. We claim that the set $\{a_1, \dots, a_n\}$ is linearly independent over K, as is the set $\{b_1, \dots, b_n\}$. Suppose not, so that $b_1, \dots, b_n$ are linearly independent, so we can write b_1 as a K-linear combination

$$b_1 = \beta_2 b_2 + \cdots + \beta_n b_n \qquad \beta_i \in K$$

Then we can write z as

$$z = \left(a_1 \otimes \sum_{i=2}^{n} \beta_i b_i \right) + \sum_{i=2}^{n} a_i \otimes b_i = \sum_{i=2}^{n} (\beta_i a_1 + a_i) \otimes b_i$$

contradicting the minimality of n from earlier. The same argument with roles reversed shows the linear independence of the a_i. Now we claim that $a_i \in Z(A)$ and $b_i \in Z(B)$ for $i = 1, \dots, n$. For any $a \in A$, since $z \in Z(A \otimes B)$, we have

$$0 = (a \otimes 1)z - z(a \otimes 1) = \sum_{i=1}^{n} (aa_i - a_i a) \otimes b_i$$

Then by linear indepence of the b_i and Lemma 2.2, the we have $aa_i - a_i a = 0$ for all i, that is, $aa_i = a_i a$ which says that $a_i \in Z(A)$ for all i. By the same argument with roles reverse, $b_i \in Z(B)$ for all i. Hence $z \in Z(A) \otimes Z(B)$. □

2.3 $A \otimes_K B$ is simple if A, B are simple and B is central

Proposition 2.4 (Tensor product of central simple algebras is simple). *Let A be a central simple K-algebra and B any K-algebra. Then any two sided ideal of $A \otimes_K B$ is of the form $A \otimes_K \mathfrak{b}$ for some two sided ideal $\mathfrak{b} \subset B$. In particular, if B is simple, then $A \otimes_K B$ is simple.*

Proof. See Theorem 2 of Rapinchuk [?] or Proposition 2.6 of Milne [?]. Both proofs are not very interesting, just technical. The flavor of the proof is very similar to that of Proposition 2.3. □

2.4 Inverse in $\mathrm{Br}(K)$ given by $[A^{\mathrm{op}}]$

Proposition 2.5 (Inverses for Brauer group). *Let A be a simple K-algebra of dimension d. Then*

$$A \otimes_K A^{\mathrm{op}} \cong \mathrm{End}_K(A) \cong M_d(K)$$

Note that these are isomorphisms of K-algebras.

Proof. For $a \in A$, define

$$L_a : A \to A \qquad x \mapsto ax$$
$$R_a : A \to A \qquad x \mapsto xa$$

Note that $L_a, R_a \in \mathrm{End}_K(A)$. Then define

$$L : A \quad \to \mathrm{End}_K(A) \qquad a \mapsto L_a$$
$$R : A^{\mathrm{op}} \to \mathrm{End}_K(A) \qquad a \mapsto R_a$$

We claim that L, R are K-algebra homomorphisms. First we verify K-linearity. Let $a \in A, \lambda \in K$.

$$L_{\lambda a} = (x \mapsto \lambda a x) = \lambda(x \mapsto ax) = \lambda L_a$$
$$R_{\lambda a} = (x \mapsto x\lambda a) = \lambda(x \mapsto xa) = \lambda R_a$$

Now we verify that they preserve multiplication. Let $a, b \in A$. We denote multiplication in A^{op} by $a \cdot b = ba$. (Adjacent letters with no symbol denotes usual multiplication in A.)

$$L_{ab} = (x \mapsto abx) = (x \mapsto ax) \circ (x \mapsto bx) = L_a L_b$$
$$R_{a \cdot b} = (x \mapsto x(a \cdot b)) = (x \mapsto xba) = (x \mapsto xa) \circ (x \mapsto xb) = R_a R_b$$

Now we note that for $a, b \in A$, L_a, R_b commutes in $\mathrm{End}_K(A)$.

$$L_a R_b(x) = L_a(bx) = abx = R_b(ax) = R_b L_a(x)$$

Thus we have a K-algebra homomorphism

$$F : A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_K(A) \qquad a \otimes b \mapsto L_a R_b = R_b L_a = (x \mapsto axb)$$

Since A is simple, so is A^{op}, so by Proposition 2.4, $A \otimes_K A^{\mathrm{op}}$ is simple. Hence since F is not the zero morphism, it is injective. But then by dimension counting, it is also surjective, so

$$A \otimes_K A^{\mathrm{op}} \cong \mathrm{End}_K(A)$$

As a K-vector space, A is just K^d, so the final isomorphism $\mathrm{End}_K(A) \cong M_d(K)$ is the usual basis-dependent isomorphism between K-linear maps $K^d \to K^d$ and $d \times d$ matrices with entries in K. □

References